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Hidden symmetry algebra and overlap coefficients for two ring-shaped potentials*

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Abstract. A new approach is proposed to general systems possessing SU(1, 1)⊕SU(1, 1) dynamical symmetry. On the base of this approach, the quadratic Hahn algebra QH(3) is shown to serve as a hidden symmetry (in both quantum and classical pictures) for two potentials generalizing the Hartmann and the oscillator ring-shaped potentials. The overlap coefficients between wavefunctions in spherical and parabolic (cylindrical) coordinates are shown to coincide with Clebsch–Gordan coefficients for SU(1, 1) algebra.

1. Introduction

Generalized ring-shaped Coulomb and oscillator potentials are described by the Hamiltonians

$$H_1 = H_0 - \alpha/r + d_1/r^2(1+\cos\theta) + d_2/r^2(1-\cos\theta) \tag{1.1a}$$

$$H_2 = H_0 + \omega^2 r^2/2 + g_1/(x^2+y^2) + g_2/z^2 \tag{1.1b}$$

where $H_0 = p^2/2$ is the free motion Hamiltonian, d_i and g_i are arbitrary positive parameters.

The potentials (1.1) were introduced (among many other ones) in [1] where the general problem of finding the potentials admitting the separation of variables in several coordinate systems was studied.

For $d_1 = d_2$ the potential (1.1*a*) was studied by Hartmann [2] to describe axial symmetric systems like ring-shaped molecules. The Hartmann potential was also intensively studied from different points of view in [2–4] and others. The potential (1.1*b*) with $g_2 = 0$ (the so-called rig-shaped oscillator potential) was studied in [5, 6]. The generalized ring-shaped potentials (1.1) with $d_1 \neq d_2$ and $g_1 g_2 \neq 0$ were studied in [7–9].

There are a number of reasons for maintaining the permanent interest of physicists in these potentials:

(i) these potentials can be used in quantum chemistry and nuclear physics for describing ring-shaped molecules and deformed nuclei;

(ii) these potentials manifest non-trivial hidden symmetry leading to degeneration of the energy spectrum;

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(iii) the Schrödinger equation for these potentials admits the separation of variables in several coordinate systems, so an interesting problem arises of how to overlap the wavefunctions in different systems;
(iv) the hidden symmetry and degeneration problem for the classical potentials (1.1) seems to be non-trivial.

Note that the problem of hidden symmetry was considered in [3–5] where SU(2) algebra (constructed from the dynamical variables of the ring-shaped potentials) was chosen to be an appropriate tool. However, the approach based on SU(2), has some difficulties. In particular, it is not clear, how to translate this approach from the quantum picture into the classical one.

Another approach to the hidden symmetry problem was proposed in [6, 10]: it was shown that the integrals of motion of the Hartmann and ring-shaped potentials form quadratic Hahn algebra QH(3) under the commutations. As we show in this paper the same QH(3) algebra remains as the hidden symmetry algebra for the generalized potentials (1.1). Moreover, this statement is valid also in the classical picture (if one replaces the commutators by Poisson brackets).

Direct calculation of the overlap coefficients (using explicit wavefunctions) for the Hartmann and ring-shaped oscillator potentials was carried out in [11, 12]. In [6, 10] it was shown that these coefficients can be obtained by a purely algebraic method based on established QH(3) symmetry.

In this paper we obtain the overlap coefficients for the generalized ring-shaped potentials (1.1).

In section 2 we propose a general scheme based on SU(1, 1)$\oplus$SU(1, 1) addition. This scheme does not depend on concrete realization of the SU(1, 1) generators. We formulate the problem of hidden symmetry for the systems admitting such a scheme. The hidden symmetry algebra appears to be the Hahn quadratic algebra QH(3), whereas the overlap coefficients are nothing else than Clebsch–Gordan coefficients (CGC) for the SU(1, 1) algebra.

In section 3 we consider a concrete realization of the abstract scheme leading to the oscillator ring-shaped potential (1.1*b*).

In section 4 we analyse another realization leading to the generalized Coulomb ring-shaped potential (1.1*a*).

In section 5 we outline the situation in terms of the classical picture for the potentials (1.1).

Note that in [7, 8] SU(1, 1) algebra was used as a dynamical symmetry generating the spectrum of the Hamiltonians (1.1). In this paper we use SU(1, 1) algebra in order to construct hidden symmetry algebra QH(3) describing the degeneration of the energy levels.

Recently, Kuznetsov [13] showed that SU(1, 1) algebra plays the crucial role in the problems of hidden symmetry and separation of variables for the Laplace-Beltrami equations. We adopt some ideas of [13] to describe the hidden symmetry for the Hamiltonians (1.1). However, our approach differs from that of [13].

2. Abstract dynamical system related to the SU(1, 1)$\oplus$SU(1, 1) scheme

Consider mutually commuting SU(1, 1) algebras defined by the standard relations

$$[A_0^{(i)}, A_\pm^{(i)}] = \pm A_\pm^{(i)}, [A_-^{(i)}, A_+^{(i)}] = 2A_0^{(i)} \qquad i = 1, 2. \tag{2.1}$$

Unitary representations of SU(1, 1) are defined by the value of the Casimir operator

$$Q = A_0^2 - A_0 - A_+A_- = a(a-1). \tag{2.2}$$

For the representations of the positive discrete series D_1^+ and D_2^+ we have

$$\begin{aligned} A_0^{(i)}|n_i, a_i\rangle &= (n_i + a_i)|n_i, a_i\rangle \\ A_-^{(i)}|n_i, a_i\rangle &= [n_i(n_i+2a_i-1)]^{1/2}|n_i-1, a_i\rangle \\ A_+^{(i)}|n_i, a_i\rangle &= [(n_i+1)(n_i+2a_i)]^{1/2}|n_i+1, a_i\rangle \end{aligned} \tag{2.3}$$

where $a_i > 0$ are the representation's parameters, $n_i = 0, 1, 2, \ldots$. One can construct the direct sum of the initial SU(1, 1) algebras

$$A_0^{(3)} = A_0^{(1)} + A_0^{(2)} \qquad A_\pm^{(3)} = A_\pm^{(1)} + A_\pm^{(2)}. \tag{2.4}$$

The connected basis $|n_3, a_3\rangle$ is defined by

$$\begin{aligned} A_0^{(3)}|n_3, a_3\rangle &= (n_3 + a_3)|n_3, a_3\rangle \\ Q_3|n_3, a_3\rangle &= a_3(a_3-1)|n_3, a_3\rangle \end{aligned} \tag{2.5}$$

Given the values a_1 and a_2, the parameter a_3 can take the discrete set of values

$$a_3 = a_1 + a_2 + p \qquad p = 0, 1, 2, \ldots. \tag{2.6}$$

The Clebsch–Gordan decomposition is written as

$$|j, a_3\rangle = \sum_{n=0}^{N} (n, a_1; N-n, a_2|j, a_3)|n, a_1\rangle \otimes |N-n, a_2\rangle \tag{2.7}$$

where

$$N = j + p. \tag{2.8}$$

The CGC have been analysed and calculated in many papers (see, for example [14–16]). We use an expression for CGC in terms of Hahn polynomials [17]

$$(n, a_1; N-n, a_2|j, a_3) = h_n w_p\, {}_3F_2\left[\begin{matrix} -n, -p, 2a_1+2a_2+p-1 \\ -N, 2a_1 \end{matrix}; 1\right] \tag{2.9}$$

where

$$h_n = \frac{(-N)_n(2a_1)_n}{n!(1-2a_2-N)_n} \tag{2.10}$$

is a normalization factor and

$$w_p = \frac{(2a_2)_N(2a_1+2a_2+2p-1)(-1)^p(-N)_p(2a_1+2a_2-1)_p(2a_1)_p}{p!(2a_1+2a_2)_N(2a_1+2a_2-1)(2a_1+2a_2+N)_p(2a_2)_p} \tag{2.11}$$

is the weight amplitude for the Hahn polynomials ${}_3F_2\,(\ldots;1)$ of the argument

$$\mu_p = (a_1 + a_2 + p)(a_1 + a_2 + p - 1) \qquad p = 0, 1, \ldots, N \tag{2.12}$$

Now we can formulate the following problem. Consider the system with the Hamiltonian

$$H = \sigma^{-1} A_0^{(3)} \tag{2.13}$$

where σ is some real constant. This means that SU(1, 1)⊕SU(1, 1) serves as dynamical symmetry for the Hamiltonian (2.13). Indeed, the operators $A_\pm^{(1)}$ and $A_\pm^{(2)}$ are the ladder operators generating the spectrum $\varepsilon=\sigma^{-1}(n_1+n_2+a_1+a_2)$ of the Hamiltonian (2.13). It is clear that the spectrum is degenerated. So one can expect the existence of integrals commuting with the Hamiltonian (2.13). In frames of our abstract scheme there are two independent integrals commuting with H:

$$K_1=A_0^{(1)}-A_0^{(2)} \tag{2.14a}$$

$$K_2=Q_3=a_1(a_1-1)+a_2(a_2-1)+2A_0^{(1)}A_0^{(2)}-A_+^{(1)}A_-^{(2)}-A_-^{(1)}A_+^{(2)}. \tag{2.14b}$$

We are seeking the hidden symmetry algebra of the system (2.13). By 'hidden symmetry' we mean some algebra constructed from the integrals K_1, K_2. Recall that for many well known problems the integrals of motion form a finite-dimensional Lie algebra (e.g. O(4) for the Coulomb problem or U(3) for the isotropic oscillator).

Do the integrals (2.14) form any algebra? The answer is positive, however the algebra is nonlinear. To see this let us introduce the third integral

$$K_3=[K_1, K_2]=2(A_-^{(1)}A_+^{(2)}-A_+^{(1)}A_-^{(2)}). \tag{2.15a}$$

Then using commutation relations (2.1) one can easily verify that on the subspace with given value ε the operators K_1, K_2, K_3 are closed in frames of quadratic algebra under the commutations:

$$[K_2, K_3]=-2(K_1K_2+K_2K_1)+4\sigma\varepsilon(a_1^2-a_1-a_2^2+a_2) \tag{2.15b}$$

$$[K_3, K_1]=-2K_1^2-4K_2+2\varepsilon^2\sigma^2+4(a_1^2-a_1+a_2^2-a_2). \tag{2.15c}$$

The commutation relations (2.15) define quadratic Hahn algebra QH(3) which was introduced and analysed in [6, 10, 18]. An attractive feature of QH(3) is that all its finite-dimensional representations can be easily constructed in analogy with three-dimensional Lie algebras.

The overlaps between eigenstates of the operators K_1 and K_2 can be expressed in terms of Hahn polynomials. On the other hand, for concrete realization (2.14) the diagonalization of the operator K_1 corresponds to choosing the unconnected basis $|n_1, a_1\rangle\otimes|n_2, a_2\rangle$ in the space of the direct sum SU(1, 1)⊕SU(1, 1), whereas diagonalization of the operator K_2 corresponds to choosing the connected basis $|n_3, a_3\rangle$ in the same space. So, in the case (2.14) the overlaps between the operators K_1 and K_2 coincide with CGC for the SU(1, 1) algebra. This explains the appearance of the Hahn polynomials in expression (2.9) for the CGC of SU(1, 1).

It is worth mentioning that the algebra QH(3) does not reduce to the SU(1, 1) algebra because the formulae (2.14) provide only one of many possible realizations of QH(3). For example, there is an analogous realization of QH(3) in terms of SU(2)⊕SU(2) generators [18] allowing explanation of the appearance of the Hahn polynomials in the expression for the CGC of SU(2).

So far, we have considered a_1 and a_2 as fixed real parameters. For concrete problems, both a_1 and a_2 may be the eigenvalues of some additional integrals commuting with K_1 and K_2 as well as with H. In these cases an additional degeneration (not related to the SU(1, 1)⊕SU(1, 1) scheme) is possible.

3. Generalized ring-shaped oscillator

Consider the following realization of the SU(1, 1) operators:

$$A_0^{(1)} = -\frac{1}{4\omega}(\partial_x^2+\partial_y^2) + \frac{\omega}{4}(x^2+y^2) + \frac{g_1}{2\omega(x^2+y^2)}$$

$$A_0^{(1)} + A_1^{(1)} = \frac{\omega}{2}(x^2+y^2)/2 \tag{3.1}$$

$$A_2^{(1)} = \mathrm{i}(x\partial_x + y\partial_y + 1)/2$$

$$A_0^{(2)} = \frac{1}{4\omega}\partial_z^2 + \frac{\omega}{4}z^2 + g_2/2\omega z^2$$

$$A_0^{(2)} + A_1^{(2)} = \omega z^2/2 \tag{3.2}$$

$$A_2^{(2)} = \mathrm{i}(z\partial_z + 1/2)/2$$

where $A_\pm = A_1 \pm \mathrm{i}A_2$ and ω, g_1, g_2 are some real positive constants.

Given the realization (3.1)–(3.2), the Casimir operators are

$$Q_1 = (m^2 + 2g_1 - 1)/4 \qquad Q_2 = g_2/2 - 3/16 \tag{3.3}$$

where m is the azimuthal quantum number, i.e. the eigenvalue of the operator $L_z = -\mathrm{i}(x\partial_y - y\partial_x)$ obviouly commuting with the SU(1, 1) generators (3.1) and (3.2).

In accordance with the approach of the previous section, let us choose the Hamiltonian to be

$$H = 2\omega A_0^{(3)} = -(\partial_x^2+\partial_y^2+\partial_z^2)/2$$
$$+\frac{\omega^2}{2}(x^2+y^2+z^2) + g_1/(x^2+y^2) + g_2/z^2 \tag{3.4}$$

The Hamiltonian (3.4) coincides with (1.1*b*) describing the ring-shaped oscillator.

Using the results of the previous section, we immediately obtain the integrals

$$K_1 = A_0^{(1)} - A_0^{(2)} = H_{xy} - H_z$$
$$K_2 = Q_3 = L^2/4 + g_1 r^2/2(x^2+y^2) + g_2 r^2/2z^2 - 3/16 \tag{3.5}$$

where by H_{xy} and H_z we denote corresponding generalized oscillator Hamiltonians.

The representation parameters a_1 and a_2 are found from (3.3):

$$a_1 = (1 + [m^2 + 2g_1]^{1/2})/2 \qquad a_2 = (2 + [1 + 8g_2]^{1/2})/4. \tag{3.6}$$

Diagonalization of the operator K_1 corresponds to separation of the variables in cylindrical coordinates whereas diagonalization of the operator K_2 corresponds to separation of the variables in spherical coordinates. According to the results of the previous section, we see that the overlaps between the wavefunctions in these coordinate systems coincide with the cgc of SU(1, 1) and are given by formulae (2.9).

The hidden symmetry algebra for the potential of generalized ring-shaped oscillator coincides with Hahn algebra QH(3) (2.15) where $\sigma = 1/2\omega$ and

$$\varepsilon = 2\omega(1 + [m^2 + 2g_1]^{1/2}/2 + [1 + 8g_2]^{1/2}/4 + N) \qquad N = 0, 1, 2, \ldots. \tag{3.7}$$

It is interesting to note that the case $g_2 = 0$ (ring-shaped oscillator [5, 6]) cannot be obtained automatically from these results by the simple procedure $g_2 \to 0$. Indeed, for $g_2 > 0$ the wavefunctions must obey the natural condition

$$\psi|_{z=0} = 0.$$

However, if $g_2=0$ there is no such restriction and in this case the representation parameter a_2 can take two values:

$$a_2=(2\pm 1)/4.$$

So for $g_2=0$ we obtain the spectrum

$$=2\omega(a_1+a_2+N)=\omega([m^2+2g_1]^{1/2}+\tilde{N}+3/2) \qquad \tilde{N}=0,1,2,\ldots. \tag{3.8}$$

The formula (3.8) coincides with that describing the spectrum of the ring-shaped oscillator [5, 6].

4. Generalized ring-shaped coulomb potential

In this section we choose the following realization of the SU(1, 1) algebra [3, 7, 8]:

$$\begin{aligned}
&A_0^{(1)}=\frac{1}{2\gamma}(\xi\partial_\xi^2+\partial_\xi-c_1\xi^{-1}-\gamma^2\xi)\\
&A_0^{(1)}+A_1^{(1)}=\gamma\xi\\
&A_2^{(1)}=\mathrm{i}(\xi\partial_\xi+1/2)
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
&A_0^{(2)}=-\frac{1}{2\gamma}(\eta\partial_\eta^2+\partial_\eta-c_2\eta^{-1}-\gamma^2\eta)\\
&A_0^{(2)}+A_1^{(2)}=\gamma\eta\\
&A_2^{(2)}=\mathrm{i}(\eta\partial_\eta+1/2)
\end{aligned} \tag{4.2}$$

where ξ and η are variables and γ, c_1, c_2 are real constants.

The SU(1, 1) Casimir operators (2.2) take the values

$$Q_1=c_1-1/4 \qquad Q_2=c_2-1/4. \tag{4.3}$$

Let us take the 'Hamiltonian'

$$\tilde{H}=2\gamma A_0^{(3)} \tag{4.4}$$

and choose ξ and η to be parabolic coordinates in three-dimensional Euclid space: $\xi=r(1+\cos\theta)$, $\eta=r(1-\cos\theta)$. Then the eigenvalue problem

$$\tilde{H}\psi=a\psi \tag{4.5}$$

is written as

$$-\tfrac{1}{2}\Delta\psi+\left[\frac{d_1}{r^2(1+\cos\theta)}+\frac{d_2}{r^2(1-\cos\theta)}-a/r\right]\psi=-2\gamma^2\psi \tag{4.6}$$

where $d_i=c_i-m^2/4$, m is the azimuthal quantum number and Δ is the ordinary Laplace operator.

Choosing $E=-2\gamma^2$ we see that the space $\{\Phi\}$ of the eigenfunction for the 'Hamiltonian' $\tilde{H}$ coincides with the space of the eigenfunctions for generalized ring-shaped Coulomb potential (1.1a) where E is its energy eigenvalue.

Applying the technique of section 2 we obtain two independent integrals

$$K_1 = A_0^{(1)} - A_0^{(2)} \qquad K_2 = Q_3. \tag{4.7}$$

The representations parameters are

$$a_{1,2} = (1 + [m^2 + 4d_{1,2}]^{1/2})/2. \tag{4.8}$$

The energy spectrum for the potential (1.1a) is obtained from the spectrum of $A_0^{(3)}$:

$$\alpha/2\gamma = a_1 + a_2 + N \qquad N = 0, 1, \ldots. \tag{4.9}$$

Taking into account (4.8), we obtain

$$E = -\frac{\alpha^2}{2(M_1/2 + M_2/2 + 1 + N)^2} \tag{4.10}$$

where

$$M_{1,2} = (m^2 + 4d_{1,2})^{1/2}. \tag{4.11}$$

The energy spectrum (4.10) for the potential (1.1a) was obtained (by another method) earlier in [19].

The hidden symmetry algebra for the potential (1.1a) obviously coincides with QH(3) (2.15) where $\sigma = 1/2\gamma$, $\varepsilon = \alpha$.

The diagonalization of the operator K_1 corresponds to separation of the variables in parabolic coordinates (K_1 is an analogue of the Runge–Lenz vector [20]), whereas diagonalization of the operator K_2 corresponds to separation of variables in spherical coordinates. The overlaps between these functions are given by the formulae (2.9) coinciding with the Clebsch–Gordan coefficients of the SU(1, 1) algebra.

On the other hand, it is well known [20] that for the ordinary Coulomb problem ($d_1 = d_2 = 0$) the overlaps between wavefunctions in spherical and parabolic coordinates coincide with CGC of the SU(2) algebra because the Coulomb potential has O(4) = SU(2)⊕SU(2) hidden symmetry. So it is somewhat surprising that for slightly different potential (1.1a) the overlaps, instead, coincide with the CGC of SU(1, 1) algebra. The reason is that the complete dynamical algebra for the Coulomb potential is O(4, 2) [21]. This algebra includes both SU(2)⊕SU(2) and SU(1, 1)⊕SU(1, 1) schemes. As a consequence, the CGC of SU(2) and SU(1, 1) algebras coincide for a_1 and a_2 being integer or half integer. The anisotropic terms $\approx d_{1,2}$ in (1,1a) destroy the O(4) symmetry, whereas SU(1, 1)⊕SU(1, 1) symmetry is preserved. This leads to the expression of the overlaps in terms of the CGC of SU(1, 1) algebra.

5. Some remarks on the classical picture

The ring-shaped potentials (1.1) in classical mechanics were investigated in [9], however, the algebra of the hidden symmetry was not found earlier.

In this section we only repeat the scheme of section 2 and show that the classical hidden symmetry algebra is again QH(3).

Let us define the classical SU(1, 1) algebra by the relations

$$(A_0, A_1) = A_2 \qquad (A_2, A_0) = A_1 \qquad (A_1, A_2) = -A_0 \tag{5.1}$$

where A_i are classical dynamical variables and $(.\,,.)$ stands for the Poisson brackets.

The Casimir element is

$$Q = A_0^2 - A_1^2 - A_2^2 = a^2. \tag{5.2}$$

In what follows we consider only the classical analogue of the positive discrete series: $a^2 > 0$, $A_0 > 0$.

Introducing two independent SU(1, 1) classical algebras with the parameters a_1 and a_2 we can define the Hamiltonian

$$H = \sigma^{-1} A_0^{(3)} = \sigma^{-1}(A_0^{(1)} + A_0^{(2)}). \tag{5.3}$$

There are two independent integrals

$$\begin{aligned} K_1 &= A_0^{(1)} - A_0^{(2)} \\ K_2 &= Q_3 = a_1^2 + a_2^2 + 2(A_0^{(1)}A_0^{(2)} - A_1^{(1)}A_1^{(2)} - A_2^{(1)}A_2^{(2)}) \end{aligned} \tag{5.4}$$

which are constants of the motion: $(H, K_1) = (H, K_2) = 0$.

It is easily verified that these integrals form the classical variant of QH(3):

$$\begin{aligned} (K_1, K_2) &= K_3 \\ (K_2, K_3) &= 4K_1K_2 + 4\varepsilon\sigma(a_2^2 - a_1^2) \\ (K_3, K_1) &= 2K_1^2 + 4K_2 - 2\varepsilon^2\sigma^2 - 4(a_1^2 + a_2^2). \end{aligned} \tag{5.5}$$

(For general questions concerning the relations between quantum and classical quadratic algebras like QH(3), see [18]).

All the results of section 3 and section 4 can be translated into the classical picture. For example, choosing the realization

$$\begin{aligned} A_0^{(1)} &= \frac{1}{4\omega}(p_x^2 + p_y^2) + \frac{\omega}{4}(x^2 + y^2) + \frac{g_1}{2\omega(x^2 + y^2)} \\ A_0^{(1)} + A_1^{(1)} &= \omega(x^2 + y^2)/2 \\ A_2^{(1)} &= (xp_x + yp_y)/2 \end{aligned} \tag{5.6}$$

$$\begin{aligned} A_0^{(2)} &= \frac{1}{4\omega}p_z^2 + \omega z^2/4 + g_2/2\omega z^2 \\ A_0^{(2)} + A_1^{(2)} &= \omega z^2/2 \\ A_2^{(2)} &= zp_z/2 \end{aligned} \tag{5.7}$$

where $(p_x, x) = (p_y, y) = (p_z, z) = 1$, we obtain the integrals for the classical ring-shaped oscillator (1.1*b*). Analogously one can obtain corresponding integrals for the (1.1*a*) potential.

Thus, the hidden symmetry algebra of the integrals for the classical ring-shaped potentials (1.1) coincides with (classical) QH(3). It would be interesting to establish a classical sense of the overlaps (CGC) (2.9).

6. Conclusion

We have shown that both (1.1*a*) and (1.1*b*) potentials have the same hiden symmetry algebra QH(3). The overlap coefficients between wavefunctions in spherical and cylindrical (parabolic) coordinates coincide with the CGC of SU(1, 1) algebra (2.9).

In the classical picture the same QH(3) algebra serves as hidden symmetry for the potentials (1.1).

It would be interesting to analyse by the proposed method other Hamiltonians having quadratic hidden symmetry [22].

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